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Liouville-Neumann Approach to the Nonperturbative Quantum Field Theory*

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Abstract

We present a nonperturbative field theoretic method based on the Liouville-Neumann (LN) equation. The LN approach provides a unified formulation of nonperturbative quantum fields and also nonequilibrium quantum fields, which makes use of mean-field type equations and whose results at the lowest level are identically the same as those of the Gaussian effective potential approach and the mean-field approach. The great advantageous point of this formulation is its readiness of applicability to time-dependent quantum systems and to finite temperature field theory, and its possibility to go beyond the Gaussian approximation.

I. INTRODUCTION

In this conference the terminology of *nonperturbative* has a diverse meaning. In a different community of physics it has been used in a different sense. Sometimes it means *topological*,

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solitonic, *exactly solvable*, or *dual*. However, in this talk we shall use the term *nonperturbative* to imply that the pertinent propagator represents a resummation of loop diagrams analogous to the Gaussian effective potential [1,2] and the mean-field approaches [3]. The Gaussian effective potential approach has already been reviewed in detail by Prof. Jae Hyung Yee [4].

Our formulation comes from the following observation that the Liouville-Neumann (LN) equation enables one to find the exact quantum states of the Schrödinger equation for both time-independent and time-dependent quantum systems. We work in the Schrödinger-picture

$$i\hbar \frac{\partial}{\partial t} \psi(t) = \hat{H} \psi(t). \tag{1}$$

It is known that any operator satisfying the following equations

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{O} + [\hat{O}, \hat{H}] - i\hbar \frac{\dot{\eta}}{\eta} \hat{O} = 0, \\ \hat{O} |\eta(t)\rangle = \eta(t) |\eta(t)\rangle \end{aligned} \tag{2}$$

provides the exact quantum states in terms of its eigenstates. In particular, when $\dot{\eta} = 0$, the above equation reduces nothing but to the LN equation

$$i\hbar \frac{\partial}{\partial t} \hat{O} + [\hat{O}, \hat{H}] = 0. \tag{3}$$

Furthermore, it is the same equation that the density operators should satisfy. All these together seem to suggest that the LN equation may allow in a unified way a formulation quite useful not only in quantum mechanics but also in quantum field theory, irrespective of their time-dependency.

In this talk we develop further the LN formulation, which was applied to a free quantum field in an expanding FRW Universe [5], so that it can be applied even to a self-interacting scalar field in (1+1) dimensions. The key idea for this extension to nonlinearity has already been manifested in a quantum Duffing oscillator [6].

II. ANHARMONIC OSCILLATORS AS TOY MODEL

To exploit the main idea but to keep complexity and computation minimal, we consider an anharmonic oscillator called Duffing oscillator as a $(0+1)$-dimensional toy model for a self-interacting quantum field in $(3+1)$ dimensions. It is just a quantum mechanical system. However, it exhibits all the important features of a non-trial quantum field. The Duffing oscillator has already been used as a toy model to test various field theoretical perturbation methods [7]. Let us now consider a time-independent quantum Duffing oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2} + \frac{m\lambda\hat{q}^4}{4}. \tag{4}$$

Here m is an apparent mass-parameter, ω^2 a frequency squared, and λ a coupling constant. In quantum field theory in the Minkowski spacetime, $m = 1$ and ω^2 is the real mass squared as a coupling constant, whereas in an expanding Universe $m = R(t)$ or $R^3(t)$, $R(t)$ being the scale factor of the Universe in (1+1) and (3+1) dimensions.

The basic building block both for the Gaussian effective potential approach and for the LN approach is to find an appropriate (optimizing) Fock space

$$\hat{a}^\dagger = u\hat{p} - m\dot{u}\hat{q}, \hat{a} = u^*\hat{p} - m\dot{u}^*\hat{q}. \tag{5}$$

We require the common commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, which leads to the boundary condition

$$\hbar m(\dot{u}^* u - \dot{u} u^*) = i. \tag{6}$$

Once given a Hamiltonian, one can evaluate the effective potential

$$V_{eff} = \min_{|\psi\rangle} \langle\psi|\hat{H}|\psi\rangle \tag{7}$$

where the variation is taken with respect to all the normalized state $\langle\psi|\psi\rangle = 1$. One additional condition one can further impose is either $\langle\psi|\hat{q}|\psi\rangle = 0$ or $\langle\psi|\hat{q}|\psi\rangle = q_c$. q_c corresponds to a classical background field of the field theory. We perform the expectation value of the Hamiltonian to find the effective potential

$$V_{eff} = \frac{m}{2}\hbar^2 \dot{u}^* \dot{u} + \frac{m\omega^2}{2}\hbar^2 u^* u + \frac{3\lambda}{4}(\hbar^2 u^* u)^2. \tag{8}$$

The problem of finding the Fock space that optimizes the effective potential can be solved by varying Ω of trial wave functions

$$\begin{aligned} u &= \frac{1}{\sqrt{2\hbar m \Omega}} e^{-i\Omega t}, (\Omega > 0) \\ u &= \frac{1}{\sqrt{-2\hbar m \Omega}} e^{i\Omega t}, (\Omega < 0). \end{aligned} \tag{9}$$

By minimizing the effective potential, ($\frac{\partial V_{eff}}{\partial \Omega} = 0$), we obtain the gap equation [8]

$$\Omega^3 - \omega^2 \Omega - \frac{3\hbar\lambda}{2m} = 0. \tag{10}$$

For the positive ω^2, we have a real postive Ω, and for the negative ω^2, we have always a real negative Ω for both a weak and a strong coupling constant λ, and a real positive Ω for a strong coupling constant λ.

In the LN formulation we wish to find the LN operators that satisfy Eq. (3). We normal order the Hamiltonian operator and regroup the quadratic $\hat{H}_2$, the quartic $\hat{H}_4$ and the constant vacuum expectation value as

$$\begin{aligned} \hat{H}_2 &= \Big[m\hbar^2 \dot{u}^* \dot{u} + m\omega^2 \hbar^2 u^* u + 3m\lambda(\hbar^2 u^* u)^2 \Big] \hat{a}^\dagger \hat{a} \\ &- \Big[\frac{m}{2}\hbar^2 \dot{u}^{*2} + \frac{m\omega^2}{2}\hbar^2 u^{*2} + \frac{3m\lambda}{2}(\hbar^2 u^* u)\hbar^2 u^{*2} \Big] \hat{a}^{\dagger 2} \\ &- \Big[\frac{m}{2}\hbar^2 \dot{u}^2 + \frac{m\omega^2}{2}\hbar^2 u^2 + \frac{3m\lambda}{2}(\hbar^2 u^* u)\hbar^2 u^2 \Big] \hat{a}^2, \\ \hat{H}_4 &= \frac{m\lambda}{4} \sum_{k=0}^{4} \binom{4}{k} \hbar^4 u^{*(4-k)} (-u)^k \hat{a}^{\dagger(4-k)} \hat{a}^k. \end{aligned} \tag{11}$$

To find the exact LN operators is difficult due to the infinite group structure of anharmonic oscillators. So we rely on the approximate LN equation

$$i\hbar \frac{\partial}{\partial t}\hat{a}^\dagger + \Big[\hat{a}^\dagger, \hat{H}_2\Big] = 0, i\hbar \frac{\partial}{\partial t}\hat{a} + \Big[\hat{a}, \hat{H}_2\Big] = 0. \tag{12}$$

The LN equation (12) leads to the mean-field equation

$$\ddot{u} + \omega^2 u + 3\lambda(\hbar^2 u^* u)u = 0. \tag{13}$$

The mean-field equation has a solution (9), which also satisfies the same gap equation (10). This means that the LN formulation at the lowest order gives rise to identically the same results as the Gaussian effective potential and the mean-field approaches. In fact, we see that the ground state wave function in the Fock space is given by

$$\Psi_{(0)}(q) = \Big(\frac{m\Omega}{\pi\hbar}\Big)^{1/4} e^{-\frac{m\Omega}{2\hbar}q^2}. \tag{14}$$

The merit of the LN formulation is that it can directly and straightforward be applied to time-dependent quantum systems, too. The only change that is to be made to the problem of a time-dependent quantum Duffing oscillator, $(m(t), \omega^2(t), \lambda(t))$, is the mean-field equation, which now reads

$$\ddot{u} + \frac{\dot{m}}{m}\dot{u} + \omega^2 u + 3\lambda(\hbar^2 u^* u)u = 0. \tag{15}$$

A unifying feature of the LN formulation is that one can define the density operator easily in terms of the creation and annihilation operators which have already been chosen to satisfy the LN equation:

$$\hat{\rho} = \frac{e^{-\beta\omega_0 \hat{a}^\dagger \hat{a}}}{\mathrm{Tr} e^{-\beta\omega_0 \hat{a}^\dagger \hat{a}}}. \tag{16}$$

In the context of quantum field theory, this implies that the LN formulation has a close connection with finite temperature field theory.

As the second case, we consider the quantum state that describes fluctuations around a classical background. The variable q is divided into

$$q = q_c + q_f. \tag{17}$$

q_c is the classical background and q_f is the quantum fluctuation. $\hat{q}_c$ and $\hat{q}_f$ commute each other. Likewise we divide the Hamiltonian into the sum of the classical background part and the fluctuation part and a perturbation:

$$\begin{aligned}\hat{H} &= \Big[\frac{\hat{p}_c^2}{2m} + \frac{m\omega^2}{2}\hat{q}_c^2 + \frac{m\lambda}{4}\hat{q}_c^4\Big] \\ &+ \Big[\frac{\hat{p}_f^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 + \frac{3m\lambda}{2}\hat{q}_c^2\hat{q}_f^2 + \frac{m\lambda}{4}\hat{q}^4\Big] \\ &+ \Big[\frac{\hat{p}_c\hat{p}_f}{m} + m\omega^2\hat{q}_c\hat{q}_f + m\lambda(\hat{q}_c^3\hat{q}_f + \hat{q}_c\hat{q}_f^3)\Big]\end{aligned} \tag{18}$$

We take the symmetric quantum state around q_c, that is,

$$\langle \hat{q} \rangle = \langle \hat{q}_c \rangle = q_c, \;\; \langle \hat{q}_f \rangle = 0. \tag{19}$$

It also holds that any odd power of $\hat{q}_f$ or $\hat{p}_f$ yields the zero-expectation value. In fact, the quantum state of the original variable q is a coherent state. We may quantize both the classical background and the fluctuation as in the case of symmetric vacuum state above. Then the change that is to be be made to Eq. (15) is the frequency, which has now a mean-field value

$$\omega_m^2 = \omega^2 + 3\lambda q_c^2. \tag{20}$$

This result can also be obtained in the mean-field approach by applying the factorization theorem to the original Hamiltonian (4)

$$\hat{q}^4 = 6\langle \hat{q}^2 \rangle \hat{q}^2 - 8\langle \hat{q}^3 \rangle \hat{q} + 6\langle \hat{q} \rangle^4 - 3\langle \hat{q}^2 \rangle^2, \tag{21}$$

and by inserting $\langle \hat{q}^2 \rangle = q_c^2$ and $\langle \hat{q} \rangle = \langle \hat{q}^3 \rangle = 0$. The time-dependent case can be treated similarly.

III. $\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4$ FIELD THEORY

To extend the LN formulation to quantum field theory, we consider a field theoretical model of a (1+1)-dimensional field. Our model has the Hamiltonian

$$H = \int dx \left[\frac{1}{2}\Pi^2(x,t) + \frac{m^2}{2}\Phi^2(x,t) + \frac{1}{2}(\nabla\Phi)^2 + \frac{\lambda}{4}\Phi^4(x,t) \right]. \tag{22}$$

The technical idea behind the extension from a quantum system to a quantum field is that the Hamiltonian can be rewritten as a sum of either decoupled or coupled anharmonic oscillators in terms of Fourier-modes. To see how this work we consider separately a free scalar field ($\lambda = 0$) and a self-interacting scalar field ($\lambda \neq 0$).

A. Free Scalar Field

We use the box normalization of the field confined into a square well of length L. We denote the orthonormal basis by

$$\xi_\alpha = \sqrt{\frac{2}{L}}\cos\Big(\frac{(2n-1)\pi x}{L}\Big) \ or \ \sqrt{\frac{2}{L}}\sin\Big(\frac{2n\pi x}{L}\Big), \tag{23}$$

where $\alpha = \frac{n\pi}{L}$ and $\sum_\alpha = \sum_{n=1}^\infty$. We decompose the field into modes

$$\Phi(x,t) = \sum_\alpha \phi_\alpha(t)\xi_\alpha(x). \tag{24}$$

The Hamiltonian is decomposed into a collection of harmonic oscillators

$$H = \sum_\alpha H_\alpha = \sum_\alpha \frac{1}{2}\pi_\alpha^2 + \frac{m^2+\alpha^2}{2}\phi_\alpha^2. \tag{25}$$

We quantize the system according to the Schrödinger-picture

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle. \tag{26}$$

The wave function as expected is easily found to be $|\Psi\rangle = \prod_\alpha |\psi_\alpha\rangle$ whose wave function for each mode obeys the Schrödinger equation with the harmonic oscillator Hamiltonian H_α. Each position operator has a Fock space representation

$$\hat{\phi}_\alpha = (-i\hbar)\Big(u_\alpha^* \hat{a}_\alpha^\dagger - u_\alpha \hat{a}_\alpha\Big). \tag{27}$$

Firstly, we assume $\langle\hat{\phi}_\alpha\rangle = 0$, which in turn implies $\langle\hat{\Phi}\rangle = 0$. The nontrivial contributions come only from even power of field operators; for instance, the correlation function is given by

$$\langle\hat{\Phi}^2\rangle = \hbar^2\sum_\alpha u_\alpha^* u_\alpha = \hbar^2\sum_{n=1}^\infty u_{\frac{n\pi}{L}}^* u_{\frac{n\pi}{L}}. \tag{28}$$

We calculate the expectation value of the Hamiltonian with respect to the vacuum state $|0,t\rangle = \prod_\alpha |0_\alpha, t\rangle$ to obtain the effective potential

$$V_{eff} = \frac{\hbar}{2}\sum_{n=1}^\infty \Big(m^2 + \Big(\frac{n\pi}{L}\Big)^2\Big)^{1/2}. \tag{29}$$

It is the sum of the vacuum fluctuations of each oscillator, which requires a renormalization.

Secondly, we consider the case of $\langle\hat{\phi}_\alpha\rangle = \phi_c$, i.e. $\langle\hat{\Phi}\rangle = \phi_c$, we divide the field into $\phi_\alpha = \phi_c + \phi_{f\alpha}$. After quantizing the fluctuations, we obtain the effective Hamiltonian

$$V_{eff} = \int dx\Big(\frac{1}{2}\pi_c^2 + \frac{m^2}{2}\phi_c^2\Big) + \frac{\hbar}{2}\sum_{n=1}^{\infty}\Big(m^2 + \big(\frac{n\pi}{L}\big)^2\Big)^{1/2}. \tag{30}$$

B. Self-Interacting Scalar Field

Firstly, we consider the case $\langle\hat{\Phi}\rangle = 0$. Recollecting that only even power of field operators contribute, we compute

$$\langle\hat{\Phi}^4\rangle = 3\Big[\sum_\alpha \hbar^2 u_\alpha^* u_\alpha\Big]^2 \tag{31}$$

The effective potential now reads that

$$V_{eff} = \frac{\hbar^2}{2}\sum_\alpha\Big[\dot{u}_\alpha^*\dot{u}_\alpha + \Big(m^2 + \alpha^2\Big)u_\alpha^* u_\alpha\Big] + \frac{3\hbar^4\lambda}{4}\Big[\sum_\alpha u_\alpha^* u_\alpha\Big]^2. \tag{32}$$

Either from the minimization of the effective potential or from the LN equation at the quadratic level (lowest order), we obtain the field equation

$$\ddot{u}_\alpha + \Big[m^2 + \big(\frac{n\pi}{L}\big)^2\Big]u_\alpha + 3\hbar^2\lambda I_0 u_\alpha = 0, \tag{33}$$

where

$$I_0 = \sum_\alpha u_\alpha^* u_\alpha = \sum_{n=1}^{\infty} u_{\frac{n\pi}{L}}^* u_{\frac{n\pi}{L}}. \tag{34}$$

Since I_0 contains an infinite quantity, it requires the renormalization of coupling constants m^2 and λ. Then the renormalized field equation should read

$$\ddot{u}_\alpha + \Big[m_R^2 + \big(\frac{n\pi}{L}\big)^2\Big]u_\alpha + 3\hbar^2\lambda_R I_{0,R} u_\alpha = 0. \tag{35}$$

We did not show the renormalization procedure in detail.

Secondly, in the case $\langle\hat{\Phi}\rangle = \phi_c$, we repeat the same procedure as for the anharmonic oscillator but keep in mind that we now treat a field rather than a finite degrees of freedom. We divide the field into a classical background field and a fluctuation field

$$\Phi(\mathbf{x},t) = \phi_c(t) + \Phi_f(\mathbf{x},t). \tag{36}$$

$\langle\hat{\Phi}\rangle = \phi_c$ and $\langle\hat{\Phi}_f\rangle = 0$ means a condensation of bosonic particles and a (collective) coherent motion of ϕ_c. We rewrite the Hamilonian density as

$$\begin{aligned}\hat{\mathcal{H}} &= \Big[\frac{1}{2}\hat{\pi}_c^2 + \frac{m^2}{2}\hat{\phi}_c^2 + \frac{\lambda}{4}\hat{\phi}_c^4\Big] \\ &+ \Big[\frac{1}{2}\hat{\Pi}_f^2 + \frac{m^2}{2}\hat{\Phi}^2 + \frac{1}{2}(\nabla\hat{\Phi}_f)^2 + \frac{3\lambda}{2}\hat{\phi}_c^2\hat{\Phi}_f^2 + \frac{\lambda}{4}\hat{\Phi}_f^4\Big] \\ &+ \Big[\hat{\pi}_c\hat{\Phi}_f + m^2\hat{\phi}_c\hat{\Phi}_f + \lambda(\hat{\phi}_c^3\hat{\Phi}_f + \hat{\phi}_c\hat{\Phi}_f^3)\Big].\end{aligned} \tag{37}$$

By quantizing the fluctuation field, we obtain the effective potential of fluctuation

$$\begin{aligned}V_{eff} = \frac{\hbar^2}{2}\sum_\alpha\Bigg[\dot{u}_\alpha^*\dot{u}_\alpha + \Big(m^2 + \alpha^2 + 3\lambda\phi_c^2\Big)u_\alpha^* u_\alpha\Bigg] \\ + \frac{3\hbar^4\lambda}{4}\Bigg[\sum_\alpha u_\alpha^* u_\alpha\Bigg]^2.\end{aligned} \tag{38}$$

It should be noted that the effective mass squared has been changed to $m^2 + 3\lambda\phi_c^2$. We find the renormalized field equation

$$\ddot{u}_\alpha + [m_R^2 + \alpha^2 + 3\lambda_R\phi_c^2]u_\alpha + 3\hbar^2\lambda_R I_{0,R} u_\alpha = 0. \tag{39}$$

C. Cosmological Model

As a time-dependent quantum system, we consider a self-interacting scalar field in an expanding FRW universe in (1+1) dimensions

$$ds^2 = -dt^2 + R^2(t)dx^2. \tag{40}$$

$R(t)$ is the scale factor of the Universe. The Hamiltonian takes the form

$$H = \int dx\Bigg[\frac{1}{2R}\Pi^2(x,t) + \frac{m^2R}{2}\Phi^2(x,t) + \frac{1}{2R}(\frac{d}{dx}\Phi)^2 + \frac{\lambda R}{4}\Phi^4(x,t)\Bigg]. \tag{41}$$

In (3+1) dimensional cosmology, Φ plays the role of both an inflaton (a classical background field) and a fluctuation. As in the Minkowski spacetime, we again divide the field into

$\Phi(\mathbf{x},t) = \phi_c(t) + \Phi_f(\mathbf{x},t)$, where $\langle\hat{\Phi}\rangle = \phi_c$ and $\langle\hat{\Phi}_f\rangle = 0$, and obtain the effective potential of fluctuation

$$V_{eff} = \frac{\hbar^2 R}{2}\sum_\alpha\Big[\dot{u}^*_\alpha\dot{u}_\alpha + \Big(m^2 + \frac{\alpha^2}{R^2} + 3\lambda\phi_c^2\Big)u^*_\alpha u_\alpha\Big] \\ + \frac{3\hbar^4\lambda R}{4}\Big[\sum_\alpha u^*_\alpha u_\alpha\Big]^2. \tag{42}$$

It should be noted that the effective mass squared again has been changed to $m^2+3\lambda\phi_c^2$. This formulation can be used to treat some cosmological issues such as preheating mechanism.

IV. BEYOND GAUSSIAN APPROXIMATION

We suggest how to go beyond the Gaussian approximation. We return to the toy model and construct the LN operators in a perturbative way

$$\hat{A}^\dagger = \hat{a}^\dagger + \sum_{n=1}^{\infty}(m\lambda)^n\hat{B}^\dagger_{2n+1},\ \ \hat{A} = \hat{a} + \sum_{n=1}^{\infty}(m\lambda)^n\hat{B}_{2n+1} \\ \hat{B}_{2n+1} = \sum_{k=0}^{2n+1} b_k^{(2n+1)}\hat{a}^{\dagger(2n+1-k)}\hat{a}^k. \tag{43}$$

The $\hat{U}_{(q+p=2)} = \hat{a}^{p\dagger}\hat{q}^q$ basis of the group SU(1,1) leads to Eq. (12). In $\hat{U}_{(p+q=3)}$, we obtain

$$\frac{\partial}{\partial t}\hat{B}_3 = i\Big(\Big[\hat{B}_3, \hat{H}_2\Big] + \Big[\hat{a}, \hat{H}_4\Big] + m\lambda\Big[\hat{B}_3, \hat{H}_4\Big]_{(3)}\Big). \tag{44}$$

This leads to an inhomogeneous equation of the form

$$\frac{d}{dt}\vec{B}(t) = im\lambda\mathbf{M}(u,u^*)\vec{B}(t) + i\vec{D}, \tag{45}$$

where $\vec{B}$ is a vector of b_k^3, $\mathbf{M}$ and $\vec{D}$ are a matrix and a vector depending on u and u^*, respectively. We just sketch the procedure. The first step is to solve Eq. (45) to find the spectrum generating operators that involve the creation and annihilation operators up to cubic terms. At the next step we use the holomorphic (Bargmann) representation of the Fock space [9]

$$\hat{a}^\dagger \to a^*, \hat{a} \to \frac{\partial}{\partial a^*}. \tag{46}$$

With respect to the inner product on the space of analytic functions of a^*

$$\langle \Psi_1 | \Psi_2 \rangle = \int \frac{da^* da}{2\pi i} \Psi_1^*(a^*) \Psi_2(a^*) e^{-a^* a}, \tag{47}$$

the number states of the Fock space are represented as $\langle a^* | n, t \rangle = \frac{a^{*n}}{\sqrt{n!}}$. Now at the first level, we may define the nonperturbative ground state by

$$\hat{A} \Psi_0^{(3)} = \left[\frac{\partial}{\partial a^*} + \sum_{k=0}^{3} b_k^{(3)} a^{*(3-k)} \Big(\frac{\partial}{\partial a^*} \Big)^k \right] \Psi_0^{(3)} = 0. \tag{48}$$

Finally, the density operator defined as

$$\hat{\rho} = \frac{e^{-\beta \Omega_0 \hat{A}^\dagger \hat{A}}}{\mathrm{Tr} e^{-\beta \Omega_0 \hat{A}^\dagger \hat{A}}} \tag{49}$$

satisfies the LN equation. It should be noted that Eq. (49) really goes beyond the quadratic order for the Gaussian-type.

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